

# NAG Fortran Library Routine Document

## G13EAF

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of **bold italicised** terms and other implementation-dependent details.

### 1 Purpose

G13EAF performs a combined measurement and time update of one iteration of the time-varying Kalman filter using a square root covariance filter.

### 2 Specification

```

SUBROUTINE G13EAF(N, M, L, A, LDS, B, STQ, Q, LDQ, C, LDM, R, S, K, H,
1          TOL, IWK, WK, IFAIL)
  INTEGER      N, M, L, LDS, LDQ, LDM, IWK(M), IFAIL
  real       A(LDS,N), B(LDS,L), Q(LDQ,*), C(LDM,N), R(LDM,M),
1          S(LDS,N), K(LDS,M), H(LDM,M), TOL, WK((N+M)*(N+M+L))
  LOGICAL     STQ

```

### 3 Description

The Kalman filter arises from the state space model given by:

$$X_{i+1} = A_i X_i + B_i W_i, \quad \text{var}(W_i) = Q_i$$

$$Y_i = C_i X_i + V_i, \quad \text{var}(V_i) = R_i$$

where  $X_i$  is the state vector of length  $n$  at time  $i$ ,  $Y_i$  is the observation vector of length  $m$  at time  $i$ , and  $W_i$  of length  $l$  and  $V_i$  of length  $m$  are the independent state noise and measurement noise respectively.

The estimate of  $X_i$  given observations  $Y_1$  to  $Y_{i-1}$  is denoted by  $\hat{X}_{i|i-1}$  with state covariance matrix  $\text{var}(\hat{X}_{i|i-1}) = P_{i|i-1} = S_i S_i^T$ , while the estimate of  $X_i$  given observations  $Y_1$  to  $Y_i$  is denoted by  $\hat{X}_{i|i}$  with covariance matrix  $\text{var}(\hat{X}_{i|i}) = P_{i|i}$ . The update of the estimate,  $\hat{X}_{i|i-1}$ , from time  $i$  to time  $(i+1)$ , is computed in two stages. First, the measurement-update is given by

$$\hat{X}_{i|i} = \hat{X}_{i|i-1} + K_i [Y_i - C_i \hat{X}_{i|i-1}] \quad (1)$$

and

$$P_{i|i} = [I - K_i C_i] P_{i|i-1} \quad (2)$$

where  $K_i = P_{i|i-1} C_i^T [C_i P_{i|i-1} C_i^T + R_i]^{-1}$  is the Kalman gain matrix. The second stage is the time-update for  $X$  which is given by

$$\hat{X}_{i+1|i} = A_i \hat{X}_{i|i} + D_i U_i \quad (3)$$

and

$$P_{i+1|i} = A_i P_{i|i} A_i^T + B_i Q_i B_i^T \quad (4)$$

where  $D_i U_i$  represents any deterministic control used.

The square root covariance filter algorithm provides a stable method for computing the Kalman gain matrix and the state covariance matrix. The algorithm can be summarized as

$$\begin{pmatrix} R_i^{1/2} & C_i S_i & 0 \\ 0 & A_i S_i & B_i Q_i^{1/2} \end{pmatrix} U = \begin{pmatrix} H_i^{1/2} & 0 & 0 \\ G_i & S_{i+1} & 0 \end{pmatrix} \quad (5)$$

where  $U$  is an orthogonal transformation triangularizing the the left-hand pre-array to produce the right-

hand post-array. The relationship between the Kalman gain matrix,  $K_i$ , and  $G_i$  is given by

$$A_i K_i = G_i \left( H_i^{1/2} \right)^{-1}.$$

G13EAF requires the input of the lower triangular Cholesky factors of the noise covariance matrices  $R_i^{1/2}$  and, optionally,  $Q_i^{1/2}$  and the lower triangular Cholesky factor of the current state covariance matrix,  $S_i$ , and returns the product of the matrices  $A_i$  and  $K_i$ ,  $A_i K_i$ , the Cholesky factor of the updated state covariance matrix  $S_{i+1}$  and the matrix  $H_i^{1/2}$  used in the computation of the likelihood for the model.

## 4 References

Vanbegin M, van Dooren P and Verhaegen M H G (1989) Algorithm 675: FORTRAN subroutines for computing the square root covariance filter and square root information filter in dense or Hessenberg forms *ACM Trans. Math. Software* **15** 243–256

Verhaegen M H G and van Dooren P (1986) Numerical aspects of different Kalman filter implementations *IEEE Trans. Auto. Contr.* **AC-31** 907–917

## 5 Parameters

- |    |                                                                                                                                                                                                            |              |
|----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------|
| 1: | N – INTEGER                                                                                                                                                                                                | <i>Input</i> |
|    | <i>On entry:</i> the size of the state vector, $n$ .                                                                                                                                                       |              |
|    | <i>Constraint:</i> $N \geq 1$ .                                                                                                                                                                            |              |
| 2: | M – INTEGER                                                                                                                                                                                                | <i>Input</i> |
|    | <i>On entry:</i> the size of the observation vector, $m$ .                                                                                                                                                 |              |
|    | <i>Constraint:</i> $M \geq 1$ .                                                                                                                                                                            |              |
| 3: | L – INTEGER                                                                                                                                                                                                | <i>Input</i> |
|    | <i>On entry:</i> the dimension of the state noise, $l$ .                                                                                                                                                   |              |
|    | <i>Constraint:</i> $L \geq 1$ .                                                                                                                                                                            |              |
| 4: | A(LDS,N) – <i>real</i> array                                                                                                                                                                               | <i>Input</i> |
|    | <i>On entry:</i> the state transition matrix, $A_i$ .                                                                                                                                                      |              |
| 5: | LDS – INTEGER                                                                                                                                                                                              | <i>Input</i> |
|    | <i>On entry:</i> the first dimension of the arrays A, B, S and K as declared in the (sub)program from which G13EAF is called.                                                                              |              |
|    | <i>Constraint:</i> $LDS \geq N$ .                                                                                                                                                                          |              |
| 6: | B(LDS,L) – <i>real</i> array                                                                                                                                                                               | <i>Input</i> |
|    | <i>On entry:</i> the noise coefficient matrix $B_i$ .                                                                                                                                                      |              |
| 7: | STQ – LOGICAL                                                                                                                                                                                              | <i>Input</i> |
|    | <i>On entry:</i> if $STQ = .TRUE.$ then the state noise covariance matrix $Q_i$ is assumed to be the identity matrix. Otherwise the lower triangular Cholesky factor, $Q_i^{1/2}$ , must be provided in Q. |              |
| 8: | Q(LDQ,*) – <i>real</i> array                                                                                                                                                                               | <i>Input</i> |
|    | <b>Note:</b> the second dimension of the array Q must be at least at least L if $STQ = .FALSE.$ and 1 if $STQ = .TRUE.$ .                                                                                  |              |

- On entry:* if STQ = .FALSE. Q must contain the lower triangular Cholesky factor of the state noise covariance matrix,  $Q_i^{1/2}$ . Otherwise Q is not referenced.
- 9: LDQ – INTEGER *Input*  
*On entry:* the first dimension of the array Q as declared in the (sub)program from which G13EAF is called.  
*Constraint:* if STQ = .FALSE.,  $LDQ \geq L$  otherwise  $LDQ \geq 1$ .
- 10: C(LDM,N) – *real* array *Input*  
*On entry:* the measurement coefficient matrix,  $C_i$ .
- 11: LDM – INTEGER *Input*  
*On entry:* the first dimension of the arrays C, R and H as declared in the (sub)program from which G13EAF is called.  
*Constraint:*  $LDM \geq M$ .
- 12: R(LDM,M) – *real* array *Input*  
*On entry:* the lower triangular Cholesky factor of the measurement noise covariance matrix  $R_i^{1/2}$ .
- 13: S(LDS,N) – *real* array *Input/Output*  
*On entry:* the lower triangular Cholesky factor of the state covariance matrix,  $S_i$ .  
*On exit:* the lower triangular Cholesky factor of the state covariance matrix,  $S_{i+1}$ .
- 14: K(LDS,M) – *real* array *Output*  
*On exit:* the Kalman gain matrix,  $K_i$ , premultiplied by the state transition matrix,  $A_i$ ,  $A_i K_i$ .
- 15: H(LDM,M) – *real* array *Output*  
*On exit:* the lower triangular matrix  $H_i^{1/2}$ .
- 16: TOL – *real* *Input*  
*On entry:* the tolerance used to test for the singularity of  $H_i^{1/2}$ . If  $0.0 \leq TOL < m^2 \times \mathbf{machine\ precision}$ , then  $m^2 \times \mathbf{machine\ precision}$  is used instead. The inverse of the condition number of  $H^{1/2}$  is estimated by a call to F07TGF (STRCON/DTRCON). If this estimate is less than TOL then  $H^{1/2}$  is assumed to be singular.  
*Suggested value:* TOL=0.0.  
*Constraint:*  $TOL \geq 0.0$ .
- 17: IWK(M) – INTEGER array *Workspace*  
18: WK((N+M)\*(N+M+L)) – *real* array *Workspace*
- 19: IFAIL – INTEGER *Input/Output*  
*On entry:* IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.  
*On exit:* IFAIL = 0 unless the routine detects an error (see Section 6).  
For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, for users not familiar with this parameter the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

## 6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

On entry, N < 1,  
 or M < 1,  
 or L < 1,  
 or LDS < N,  
 or LDM < M,  
 or STQ = .TRUE. and LDQ < 1,  
 or STQ = .FALSE. and LDQ < L,  
 or TOL < 0.0.

IFAIL = 2

The matrix  $H_i^{1/2}$  is singular.

## 7 Accuracy

The use of the square root algorithm improves the stability of the computations as compared with the direct coding of the Kalman filter. The accuracy will depend on the model.

## 8 Further Comments

For models with time-invariant  $A, B$  and  $C$ , G13EBF can be used.

The estimate of the state vector  $\hat{X}_{i+1|i}$  can be computed from  $\hat{X}_{i|i-1}$  by

$$\hat{X}_{i+1|i} = A_i \hat{X}_{i|i-1} + AK_i r_i$$

where

$$r_i = Y_i - C_i \hat{X}_{i|i-1}$$

are the independent one step prediction residuals. The required matrix-vector multiplications can be performed by F06PAF (SGEMV/DGEMV).

If  $W_i$  and  $V_i$  are independent multivariate Normal variates then the log-likelihood for observations  $i = 1, 2, \dots, t$  is given by

$$l(\theta) = \kappa - \frac{1}{2} \sum_{i=1}^t \ln(\det(H_i)) - \frac{1}{2} \sum_{i=1}^t (Y_i - C_i X_{i|i-1})^T H_i^{-1} (Y_i - C_i X_{i|i-1})$$

where  $\kappa$  is a constant.

The Cholesky factors of the covariance matrices can be computed using F07FDF (SPOTRF/DPOTRF).

Note that the model

$$X_{i+1} = A_i X_i + W_i, \quad \text{var}(W_i) = Q_i$$

$$Y_i = C_i X_i + V_i, \quad \text{var}(V_i) = R_i$$

can be specified either with B set to the identity matrix and STQ=.FALSE. and the matrix  $Q^{1/2}$  input in Q or with STQ = .TRUE. and B set to  $Q^{1/2}$ .

The algorithm requires  $\frac{7}{6}n^3 + n^2(\frac{5}{2}m + l) + n(\frac{1}{2}l^2 + m^2)$  operations and is backward stable (see Verhaegen and van Dooren (1986)).

## 9 Example

The example program first inputs the number of updates to be computed and the problem sizes. The initial state vector and state covariance matrix are input followed by the model matrices  $A_i, B_i, C_i, R_i$  and optionally  $Q_i$ . The Cholesky factors of the covariance matrices can be computed if required. The model matrices can be input at each update or only once at the first step. At each update the observed values are input and the residuals are computed and printed and the estimate of the state vector,  $\hat{X}_{i|i-1}$ , and the deviance are updated. The deviance is  $-2 \times \log$ -likelihood ignoring the constant. After the final update the state covariance matrix is computed from S and printed along with final estimate of the state vector and the value of the deviance.

The data is for a two-dimensional time series to which a VARMA(1,1) has been fitted. For the specification of a VARMA model as a state space model see the Chapter Introduction. The initial value of  $P, P_0$ , is the solution to

$$P_0 = A_1 P_0 A_1^T + B_1 Q_1 B_1^T.$$

For convenience, the mean of each series is input before the first update and subtracted from the observations before the measurement update is computed.

### 9.1 Program Text

**Note:** the listing of the example program presented below uses *bold italicised* terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*      G13EAF Example Program Text
*      Mark 17 Release. NAG Copyright 1995.
*      .. Parameters ..
INTEGER          NIN, NOUT
PARAMETER       (NIN=5,NOUT=6)
INTEGER          NMAX, MMAX, LMAX
PARAMETER       (NMAX=4,MMAX=2,LMAX=2)
*      .. Local Scalars ..
real           DEV, TOL
INTEGER          I, IFAIL, INFO, ISTEP, J, L, LDM, LDQ, LDS, M, N,
+              NCALL
LOGICAL          CONST, FULL, STQ
*      .. Local Arrays ..
real           A(NMAX,NMAX), AX(NMAX), B(NMAX,LMAX),
+              C(MMAX,NMAX), H(MMAX,MMAX), K(NMAX,MMAX),
+              P(NMAX,NMAX), Q(LMAX,LMAX), R(MMAX,MMAX),
+              S(NMAX,NMAX), WK((NMAX+MMAX)*(NMAX+MMAX+LMAX)),
+              X(NMAX), Y(MMAX), YMEAN(MMAX)
INTEGER          IWK(MMAX)
*      .. External Functions ..
real           sdot
EXTERNAL        sdot
*      .. External Subroutines ..
EXTERNAL        saxpy, scopy, sgemv, spotrf, strmv, strsv, G13EAF
*      .. Intrinsic Functions ..
INTRINSIC       LOG
*      .. Executable Statements ..
WRITE (NOUT,*) 'G13EAF Example Program Results'
*      Skip heading in data file
READ (NIN,*)
READ (NIN,*) NCALL, N, M, L, STQ, FULL, CONST
IF (N.LE.NMAX .AND. M.LE.MMAX .AND. L.LE.LMAX) THEN
  LDS = NMAX
  LDM = MMAX
  LDQ = LMAX
  READ (NIN,*) ((S(I,J),J=1,N),I=1,N)
  IF (FULL) THEN
    CALL spotrf('L',N,S,LDS,INFO)
    IF (INFO.GT.0) THEN
      WRITE (NOUT,*) ' S not positive definite'
      GO TO 100
    END IF
  END IF
```

```

END IF
READ (NIN,*) (X(I),I=1,N)
READ (NIN,*) (YMEAN(I),I=1,M)
TOL = 0.0e0
DEV = 0.0e0
WRITE (NOUT,*)
WRITE (NOUT,*) '          Residuals'
WRITE (NOUT,*)
*
*   Loop through data
*
DO 40 ISTEP = 1, NCALL
  IF ( .NOT. CONST .OR. ISTEP.EQ.1) THEN
    READ (NIN,*) ((A(I,J),J=1,N),I=1,N)
    READ (NIN,*) ((B(I,J),J=1,L),I=1,N)
    READ (NIN,*) ((C(I,J),J=1,N),I=1,M)
    READ (NIN,*) ((R(I,J),J=1,M),I=1,M)
    IF (FULL .AND. R(1,1).NE.0.0e0) THEN
      CALL spotrf('L',M,R,LDM,INFO)
      IF (INFO.GT.0) THEN
        WRITE (NOUT,*) ' R not positive definite'
        GO TO 100
      END IF
    END IF
    IF ( .NOT. STQ) THEN
      READ (NIN,*) ((Q(I,J),J=1,L),I=1,L)
      IF (FULL) THEN
        CALL spotrf('L',L,Q,LDQ,INFO)
        IF (INFO.GT.0) THEN
          WRITE (NOUT,*) ' Q not positive definite'
          GO TO 100
        END IF
      END IF
    END IF
  END IF
  IFAIL = 0
*
*   CALL G13EAF(N,M,L,A,LDS,B,STQ,Q,LDQ,C,LDM,R,S,K,H,TOL,IWK,
+         WK,IFAIL)
*
  READ (NIN,*) (Y(I),I=1,M)
  CALL saxpy(M,-1.0e0,YMEAN,1,Y,1)
*
*   Perform time and measurement update
*
  CALL sgemv('N',M,N,-1.0e0,C,LDM,X,1,1.0e0,Y,1)
  WRITE (NOUT,99999) (Y(I),I=1,M)
  CALL sgemv('N',N,N,1.0e0,A,LDS,X,1,0.0e0,AX,1)
  CALL sgemv('N',N,M,1.0e0,K,LDS,Y,1,1.0e0,AX,1)
  CALL scopy(N,AX,1,X,1)
*
*   Update loglikelihood
*
  CALL strsv('L','N','N',M,H,LDM,Y,1)
  DEV = DEV + sdot(M,Y,1,Y,1)
  DO 20 I = 1, M
    DEV = DEV + 2.0e0*LOG(H(I,I))
  20 CONTINUE
  40 CONTINUE
*
*   Compute P from S
*
  DO 60 I = 1, N
    CALL scopy(I,S(I,1),LDS,P(1,I),1)
    CALL strmv('L','N','N',I,S,LDS,P(1,I),1)
    CALL scopy(I-1,P(1,I),1,P(I,1),LDS)
  60 CONTINUE
  WRITE (NOUT,*)
  WRITE (NOUT,*) ' Final X(I+1:I) '
  WRITE (NOUT,*)
  WRITE (NOUT,99999) (X(J),J=1,N)

```

```

      WRITE (NOUT,*)
      WRITE (NOUT,*) ' Final Value of P'
      WRITE (NOUT,*)
      DO 80 I = 1, N
        WRITE (NOUT,99999) (P(I,J),J=1,I)
80    CONTINUE
      WRITE (NOUT,*)
      WRITE (NOUT,99998) ' Deviance = ', DEV
    END IF
100 CONTINUE
    STOP
*
99999 FORMAT (6F12.4)
99998 FORMAT (A,e13.4)
    END

```

## 9.2 Program Data

G13EAF Example Program Data

48 4 2 2 F T T

```

8.2068  2.0599  1.4807  0.3627
2.0599  7.9645  0.9703  0.2136
1.4807  0.9703  0.9253  0.2236
0.3627  0.2136  0.2236  0.0542

0.000  0.000  0.000  0.000

4.404  7.991

0.607 -0.033  1.000  0.000
0.000  0.543  0.000  1.000
0.000  0.000  0.000  0.000
0.000  0.000  0.000  0.000

1.000  0.000
0.000  1.000
0.543  0.125
0.134  0.026

1.000  0.000  0.000  0.000
0.000  1.000  0.000  0.000

0.000  0.000
0.000  0.000

2.598  0.560
0.560  5.330

-1.490  7.340
-1.620  6.350
5.200  6.960
6.230  8.540
6.210  6.620
5.860  4.970
4.090  4.550
3.180  4.810
2.620  4.750
1.490  4.760
1.170  10.880
0.850  10.010
-0.350  11.620
0.240  10.360
2.440  6.400
2.580  6.240
2.040  7.930
0.400  4.040
2.260  3.730
3.340  5.600

```

```

5.090  5.350
5.000  6.810
4.780  8.270
4.110  7.680
3.450  6.650
1.650  6.080
1.290  10.250
4.090  9.140
6.320  17.750
7.500  13.300
3.890  9.630
1.580  6.800
5.210  4.080
5.250  5.060
4.930  4.940
7.380  6.650
5.870  7.940
5.810  10.760
9.680  11.890
9.070  5.850
7.290  9.010
7.840  7.500
7.550  10.020
7.320  10.380
7.970  8.150
7.760  8.370
7.000  10.730
8.350  12.140

```

### 9.3 Program Results

G13EAF Example Program Results

Residuals

```

-5.8940    -0.6510
-1.4710    -1.0407
 5.1658     0.0447
-1.3280     0.4580
 1.3652    -1.5066
-0.2337    -2.4192
-0.8685    -1.7065
-0.4624    -1.1519
-0.7510    -1.4218
-1.3526    -1.3335
-0.6707     4.8593
-1.7389     0.4138
-1.6376     2.7549
-0.6137     0.5463
 0.9067    -2.8093
-0.8255    -0.9355
-0.7494     1.0247
-2.2922    -3.8441
 1.8812    -1.7085
-0.7112    -0.2849
 1.6747    -1.2400
-0.6619     0.0609
 0.3271     1.0074
-0.8165    -0.5325
-0.2759    -1.0489
-1.9383    -1.1186
-0.3131     3.5855
 1.3726    -0.1289
 1.4153     8.9545
 0.3672    -0.4126
-2.3659    -1.2823
-1.0130    -1.7306
 3.2472    -3.0836
-1.1501    -1.1623
 0.6855    -1.2751

```



2.3432	0.2570
-1.6892	0.3565
1.3871	3.0138
3.3840	2.1312
-0.5118	-4.7670
0.8569	2.3741
0.9558	-1.2209
0.6778	2.1993
0.4304	1.1393
1.4987	-1.2255
0.5361	0.1237
0.2649	2.4582
2.0095	2.5623

Final X(I+1:I)

3.6698	2.5888	0.0000	0.0000
--------	--------	--------	--------

Final Value of P

2.5980			
0.5600	5.3300		
1.4807	0.9703	0.9253	
0.3627	0.2136	0.2236	0.0542

Deviance = 0.2229E+03

---